M-math 1st year Mid Semester Exam
Subject : Functional Analysis

Time : 3 hours Max.Marks 100.

1. Let $(H, <,>)$ be a Hilbert space. A map $B : H \times H \to \mathbb{C}$ is said to be sesquilinear iff it is linear in the first variable and conjugate linear in the second variable. Define

$$\|B\| := \sup_{\|x\|\leq 1, \|y\|\leq 1} |B(x, y)|.$$

Say that B is bounded if $\|B\| < \infty$; is symmetric if $B(x, y) = \overline{B(y, x)}$; is non negative if $B(x, x) \geq 0$; is definite if $B(x, x) = 0$ implies $x = 0$.

a). If B is bounded, show that there exists a unique bounded linear operator A on H such that $B(x, y) = < x, Ay >$ and that $\|B\| = \|A\|$. (15)

b) Verify that if B is symmetric, non negative and definite then B defines an inner product on H. (10)

c) If B is symmetric and non negative , show that the Schwartz inequality holds : for every $x, y \in H$,

$$|B(x, y)|^2 \leq B(x, x)B(y, y).$$

(15)

2. Let M_n be the space of $n \times n$ matrices over $\mathbb{C}$, considered as bounded linear operators on $\mathbb{C}^n$. let $GL_n(\mathbb{C})$ be the group of $n \times n$ invertible matrices and let U be an open subset of $GL_n(\mathbb{C}) \subset M_n$. Define $J : U \to M_n$ by $J(A) := A^{-1}$. Show that J is Fréchet differentiable at all $A \in U$ and that if $H \in M_n$ then $J'(A)H = -A^{-1}HA^{-1}$ (15)

3. Let $(V, \|.\|)$ be a Banach space and $X = C([0, 1], V)$ the space of continuous functions from $[0, 1]$ with values in V. For $f \in X$,define $\|f\|_X := \sup_{t\in[0,1]} \|f(t)\|$. Show that $\|.\|_X$ is well defined and is a norm on X. Show that $(X, \|.\|_X)$ is a Banach space. (15)

4. Let V be a normed linear space and V^* its dual. For any subspace $Z \subset V^*$ and $f \in V^*$ define $d(f, Z) := \inf\limits_{g \in Z} \|f - g\|_{V^*}$. Let $W \subset V$ be a subspace and define $W^{\perp} := \{g \in V^* : g(x) = 0 \ \ \forall x \in W\}$.

a) Show that $W^{\perp}$ is a closed subspace of V^*. (10).

b) Let $f \in V^*$. Show that $d(f, W^{\perp}) = \|f_0\|_{W^*}$, where $f_0 \in W^*$ is the restriction of $f \in V^*$ to W. (15)

5. Let V be a normed linear space. Show that finite dimensional subspaces of V can be complemented i.e. if $W \subseteq V$ is finite dimensional , then there exists a closed subspace $Z \subseteq V$ such that $V = W \bigoplus Z$. (15)